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# The Fundamental Theorem of Best Approximation for Logarithmic and Exponential Orders

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It has become customary to understand by "the fundamental theorem of best approximation" an equivalence theorem which connects assertions of Jackson and Bernstein type on the rate of best approximation with those of Zamansky type for the derivatives of the approximating functions and with those of Stečkin type on simultaneous approximation of a function and its derivative. The first theorem of this sort in general Banach spaces was given by Butzer and Scherer [2]. The articles [3–6, 8, 9] have studied the phenomena which arise when this theorem (and related results of classical approximation theory) are extended to exponential orders of approximation, i.e., to rates $\mathcal{O}(1/\varphi(n))$, $n \to \infty$, where $\varphi(n)$ increases more rapidly than $n^\tau$ for each $\tau > 0$. For such orders, the validity of the converse of Zamansky's theorem requires an additional condition on the "distance" between the two orders involved. This condition cannot be removed entirely since, otherwise, there exist functions whose elements of best approximation satisfy an improved Zamansky-type estimate as well as others for which the straightforward analogue of the classical Zamansky inequality is already best possible, as has been shown in [8, 9].

As has been noted in [1], similar effects may also arise at the other end of the scale, i.e., for very slow rates of approximation. Our first objective in this paper is to extend the fundamental theorem to such "logarithmic" orders. This will be done in Theorem 2 below. At the same time the results of [5] on exponential and classical orders will be refined by working with a class $\Omega$ of orders whose elements, rather than those of the class $\Phi$ dealt with in [5], do not have to satisfy conditions upon their second and third derivatives, so that $\Omega$ is a grid of orders which is not only larger than $\Phi$ but also finer. Again the Zamansky-type inequality turns out to play a special role, the additional requirement being now condition $(\gamma)$ of Theorem 2, so that just those situations become the most interesting ones where $(\gamma)$ does not hold. Such

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situations are treated in Theorem 3 below, which shows that then an improved Zamansky-type estimate may hold, and in its Corollary, which gives more precise information about the necessity of condition $(\gamma)$. On the other hand, even if there holds an improved Zamansky-type estimate for particular functions $f$, there may be other $f$'s which have the same order of best approximation but do not admit any improvement. This is shown by an example in Remark 5.

Finally we investigate the question as to which extent an improvement of a Zamansky-type inequality can go in general. Theorem 4 gives the minimal rate of increase of the "derivatives" of the elements of best approximation in a Banach space (the minimality not being restricted to our grid $\Omega$ of orders) and, since it exhibits the same factor of improvement as the one obtained in Theorem 3, it also implies that the rate of increase there is sharp.

## 1. Preliminaries

By an order of approximation we mean an element $\varphi$ of the set

$$\begin{aligned}\Omega = \{\varphi(x);\ &\varphi\colon [x_\varphi, \infty) \to (0, \infty) \text{ for some } x_\varphi \geqslant 0,\ \lim_{x\to\infty} \varphi(x) = +\infty,\\ &\varphi \in C^1(x_\varphi, \infty),\ \varphi(x) = e^{g(x)},\ g'(x) > 0\ \ \forall x > x_\varphi,\\ &\limsup_{x\to\infty} g'(x) < \infty\}. \end{aligned} \tag{1.1}$$

Here $C^1(x_\varphi, \infty)$ denotes te set of functions which have a continuous derivative on $(x_\varphi, \infty)$. Thus $\Omega$ contains functions $\varphi$ with arbitrarily low rate of increase, for example,

$$\varphi_1(x) = l_r(x), \qquad r \in \mathbf{N} = \{1, 2,\ldots\},$$

where

$$l_1(x) = \log x, \qquad l_r(x) = (l_{r-1}(x)) \tag{1.2}$$

as well as classical rates like $\varphi_2(x) = x^r$, $r > 0$, and all the exponential rates considered in [5], e.g.,

$$\varphi_3(x) = x^{(\log x)^{\beta-1}}, \qquad \beta > 1; \qquad \varphi_4(x) = \exp(x^\tau), \qquad 0 < \tau \leqslant 1.$$

Also the product of any finite number of elements of $\Omega$ belongs to $\Omega$, but $\exp(x^\tau) \notin \Omega$ for $\tau > 1$ (cf. (1.3) below). We collect some elementary properties of the class $\Omega$ in the following

Lemma 1. (a) *For each* $\varphi \in \Omega$, *the inverse function* $\varphi^{-1}(x)$ *exists,* *belongs to* $C^1(\varphi(x_\varphi), \infty)$, *and increases to* $+\infty$ *as* $x \to +\infty$. *Moreover, there exists some constant* $C = C_\varphi$ *such that*

$$\varphi(x) = \mathcal{O}(e^{Cx}), \qquad x \to \infty. \tag{1.3}$$

(b) *For each* $\varphi \in \Omega$

$$\varphi(x+1)/\varphi(x) = \mathcal{O}(1), \qquad x \to \infty. \tag{1.4}$$

(c) *If* $\varphi, \varphi^* \in \Omega$ *(with* $g^*(x) = \log \varphi^*(x)$*) and*

$$\liminf_{x \to \infty} (g^{*\prime}(x)/g'(x)) > 1 \tag{1.5}$$

*then the quotient* $\varphi^*(x)/\varphi(x)$ *again belongs to* $\Omega$.

*Proof.* Assertion (a) is trivial in view of (1.1). As for (b), the mean value theorem may be applied to $g(x)$ to obtain a $\theta = \theta(x) \in (0, 1)$ such that

$$\varphi(x+1)/\varphi(x) = \exp(g(x+1) - g(x)) = \exp g'(x+\theta), \qquad x > x_\varphi,$$

which implies (1.4), in view of (1.1). Concerning (c), let $\varphi$, $\varphi^* \in \Omega$ and set $x_0 = \max(x_\varphi, x_{\varphi^*})$, $\Phi(x) = \varphi^*(x)/\varphi(x)$. Then obviously $\Phi \in C^1(x_0, \infty)$ and $\limsup_{x\to\infty}(\log \Phi(x))' < \infty$. By (1.5), there exist constants $x_\Phi \geqslant x_0$, $C > 1$ such that

$$g^{*\prime}(x) \geqslant Cg'(x), \qquad x \geqslant x_\Phi, \tag{1.6}$$

whence $(\log \Phi(x))' = g^{*\prime}(x) - g'(x) > 0$ for each $x \geqslant x_\Phi$. Integrating oer (1.6) from $x_\Phi$ to $x$ we have

$$\begin{aligned}\varphi^*(x)/\varphi^*(x_\Phi) &= \exp\{g^*(x) - g^*(x_\Phi)\} \geqslant \exp\{C(g(x) - g(x_\Phi))\}\\ &= \{\varphi(x)/\varphi(x_\Phi)\}^C,\end{aligned}$$

whence

$$\varphi^*(x)/\varphi(x) \geqslant \{\varphi^*(x_\Phi)(\varphi(x_\Phi))^C\}\, \varphi(x)^{C-1}, \qquad x \geqslant x_\Phi,$$

which implies $\lim_{x\to\infty} \Phi(x) = +\infty$, since $C > 1$. Thus $\Phi$ satisfies the conditions (1.1) and the proof is complete.

Of course, (1.5) is just a sufficient condition for $\Phi \in \Omega$ but not a necessary one. For example, among the pairs $(\varphi^*, \varphi)$ for which $\liminf_{x\to\infty} g^{*\prime}(x)/g'(x) = 1$ there are some with $\Phi \in \Omega$ (e.g., $\varphi^*(x) = x \log x$, $\varphi(x) = x$) and others with $\Phi \notin \Phi$ (e.g., $\varphi^*(x) = 2x$, $\varphi(x) = x$).

To each $\varphi \in \Omega$ we now associate a sequence of step functions $\beta(n, x)$ which will be used for the telescoping arguments in the proofs of Propositions 1, 2 and Theorems 1, 4. Let $\varphi \in \Omega$ with $g$, $x_\varphi$ as given by (1.1) and let $n_0$ denote an arbitrary integer $>x_\varphi$. Denoting by $\mathbf{P}$ the set of non-negative integers and by $[a]$ the largest integer $\leqslant a \in \mathbf{R}$ we set

$$j_n = j_n(\varphi, n_0) = [g(n) - g(n_0)], \qquad n \geqslant n_0, \tag{1.7}$$

and define for each $n \geqslant n_0$ the step function $\beta_{\varphi,n_0}(n, x)$ on $[0, \infty)$ by

$$\beta(n, x) = \beta_{\varphi,n_0}(n, x) = \varphi^{-1}(\varphi(n)\, e^{x - j_n}), \qquad x \geqslant 0. \tag{1.8}$$

Obviously $\beta(n, x)$ is well defined for each $x \geqslant 0$, $n \geqslant n_0$ (cf. also the proof of Lemma 2).

The rate of increase of $\beta(n, x)$ will be large when $\varphi$ increases slowly and vice versa. For example, if $\varphi(x) = e^x$, $n_0 = 1$, we have $\beta(n, x) = x + 1$ for each $n \in \mathbf{N}$, $x \geqslant 0$; if $\varphi(x) = x$, $n_0 = 1$, we have $\beta(n, x) = n \exp(x - [\log n])$, thus

$$e^x \leqslant \beta(n, x) < e^{x+1}, \qquad x \geqslant 0,\ n \in \mathbf{N};$$

and if $\varphi(x) = \log x$, $n_0 = 2$, we have $\beta(n, x) = \exp\{(\log n) \exp(x - [\log \log n - \log \log 2])\}$, thus

$$2^{\exp x} \leqslant \beta(n, x) < 2^{\exp(x+1)}, \qquad x \geqslant 0,\ n \in \mathbf{N},\ n \geqslant 2.$$

The following three lemmas deal with properties of $\beta(n, x)$.

LEMMA 2. *Let* $\varphi \in \Omega$, $n_0 \in \mathbf{N}$, $n_0 > x_\varphi$ and let $j_n$, $\beta(n, x)$ be defined by (1.7), (1.8). For each $n \in \mathbf{N}$, $n \geqslant n_0$ one has

(a) $j_n \geqslant 0$;

(b) $\beta(n, x)$ is a positive, strictly increasing function of $x$ on $[0, \infty)$, belongs to $C^1(0, \infty)$, and satisfies

$$\beta'(n, x) = 1/g'(\beta(n, x)), \qquad x \geqslant 0; \tag{1.9}$$

$$n_0 \leqslant \beta(n, 0) < \varphi^{-1}(e\varphi(n_0)); \tag{1.10}$$

$$\beta(n, j_n) = n; \tag{1.11}$$

$$\lim_{x \to \infty} \beta(n, x) = +\infty. \tag{1.12}$$

*Proof.* Assertion (a) is a trivial consequence of (1.7), (1.1), and also (1.11), (1.12) are obvious. By (1.7),

$$\varphi(n)\, e^{-j_n} \geqslant \varphi(n_0), \qquad n \geqslant n_0, \tag{1.13}$$

hence $\beta(n, 0) \geqslant n_0$, and $\beta(n, x)$ is well defined for each $x \geqslant 0$ in view of Lemma 1(a), as well as positive, strictly increasing and differentiable there. Also (1.9) follows readily, and, by (1.7), one has

$$\beta(n, 0) < \varphi^{-1}(\varphi(n) \exp\{g(n_0) - g(n) + 1\}) = \varphi^{-1}(e\varphi(n_0)),$$

which proves (1.10).

LEMMA 3. *Let* $\varphi, \varphi^* \in \Omega$ *with* $\varphi^*(x) = e^{g^*(x)}$, $n_0 \in \mathbf{N}$, $n_0 > \max(x_\varphi, x_{\varphi^*}) + 1$ *and let* $j_n = j_n(\varphi, n_0)$, $\beta(n, x) = \beta_{\varphi, n_0}(n, x)$ *be associated to* $\varphi$ *by* (1.7), (1.8).

(a) *If*

$$\limsup_{x \to \infty} (g^{*\prime}(x)/g'(x)) < \infty \tag{1.14}$$

*then there is a constant* $C$, *independent of* $j$ *and* $n$, *such that*

$$\varphi^*([\beta(n, j+1)])/\varphi^*([\beta(n, j)]) \leqslant C, \qquad j \in \mathbf{P},\ n \geqslant n_0. \tag{1.15}$$

(b) *If*

$$\liminf_{x \to \infty} (g^{*\prime}(x)/g'(x)) > 0 \tag{1.16}$$

*then there are constants* $C$, *independent of* $n$, *such that*

$$\sum_{j=0}^{j_n} \varphi^*([\beta(n, j)]) \leqslant C\varphi^*(n), \qquad n \geqslant n_0, \tag{1.17}$$

$$\sum_{j=j_n}^{\infty} \{\varphi^*([\beta(n, j+1)])\}^{-1} \leqslant C\{\varphi^*(n)\}^{-1}, \qquad n \geqslant n_0. \tag{1.18}$$

*Proof.* (a) By Lemma 2(b), one has, for each $n \geqslant n_0$ and $j \in \mathbf{P}$,

$$\beta(n, j) \geqslant [\beta(n, j)] \geqslant [\beta(n, 0)] > \beta(n, 0) - 1 \geqslant n_0 - 1 > x_{\varphi^*}, \tag{1.19}$$

so that $\varphi^*(\beta(n, j))$, $\varphi^*([\beta(n, j)])$ and $\varphi^*(\beta(n, j) - 1)$ are well defined for these $n$ and $j$. By (1.4), there is a constant $C$ such that $\varphi^*(x)/\varphi^*(x-1) \leqslant C$ for each $x \geqslant n_0 - 1$, whence, using the monotonicity of $\varphi^*$,

$$\begin{aligned} \varphi^*(\beta(n, j))/\varphi^*([\beta(n, j)]) &< \varphi^*(\beta(n, j))/\varphi^*(\beta(n, j) - 1) \\ &\leqslant C, \qquad n \geqslant n_0,\ j \in \mathbf{P}. \end{aligned} \tag{1.20}$$

Moreover, there is a constant $C$ such that

$$\varphi^*(\beta(n, j+1))/\varphi^*(\beta(n, j)) \leqslant C, \qquad n \geqslant n_0,\ j \in \mathbf{P}. \tag{1.21}$$

Indeed, in view of (1.19) and Lemma 2(b), for each $n \geqslant n_0$ the mean value theorem may be applied to $g^*(\beta(n, x))$ as a function of $x$, to obtain numbers $\xi = \xi(n, j) \in (j, j+1)$ such that (cf. (1.9))

$$\begin{aligned} g^*(\beta(n, j+1)) - g^*(\beta(n, j)) &= g^{*\prime}(\beta(n, \xi))\, \beta'(n, \xi) \\ &= g^{*\prime}(\beta(n, \xi))/g'(\beta(n, \xi)) \end{aligned} \tag{1.22}$$

for each $j \in \mathbf{P}$, $n \geqslant n_0$. By (1.14) and the continuity of $g^{*\prime}(x)/g'(x)$ on $[n_0, \infty)$, there is $C$ such that $g^{*\prime}(x)/g'(x) < C$ for each $x \geqslant n_0$. By (1.19) and the monotonicity of $\beta(n, x)$ (Lemma 2(b)) we have $\beta(n, \xi) > \beta(n, j) \geqslant n_0$ for each $n \geqslant n_0$, $j \in \mathbf{P}$. Thus the right-hand side of (1.22) is less than $C$ for these $n$, $j$, and (1.21) follows.

Now (1.20), (1.21) imply (1.15) since

$$\frac{\varphi^*([\beta(n, j+1)])}{\varphi^*([\beta(n, j)])} \leqslant \frac{\varphi^*(\beta(n, j+1))}{\varphi^*(\beta(n, j))} \frac{\varphi^*(\beta(n, j))}{\varphi^*([\beta(n, j)])}.$$

(b) By (1.19), $\varphi^*([\beta(n, j)])$ is well defined and the sum in (1.17) is non-void since $j_n \geqslant 0$ for each $n \geqslant n_0$, $j \in \mathbf{P}$. Using the monotonicity of $\varphi^*(x)$ and $\beta(n, x)$ we have

$$\sum_{j=0}^{j_n} \varphi^*([\beta(n, j)]) \leqslant \sum_{j=0}^{j_n} \varphi^*(\beta(n, j)) < \int_0^n \varphi^*(\beta(n, x))\, dx, \qquad n \geqslant n_0.$$

Setting $t = \beta(n, x)$ it follows by (1.10), (1.11), (1.9) that

$$\int_0^n \varphi^*(\beta(n, x))\, dx \leqslant \int_{n_0}^n \varphi^*(t)\, g'(t)\, dt = \int_{n_0}^n \varphi^{*\prime}(t)(g'(t)/g^{*\prime}(t))\, dt.$$

By (1.16) and the continuity of $g^{*\prime}(x)/g'(x)$ on $[n_0, \infty)$ there is $C$ such that

$$g'(t)/g^{*\prime}(t) < C, \qquad t \geqslant n_0, \tag{1.23}$$

and hence

$$\sum_{j=0}^{j_n} \varphi^*([\beta(n, j)]) \leqslant C \int_{n_0}^n \varphi^{*\prime}(t)\, dt = C(\varphi^*(n) - \varphi^*(n_0)),$$

which proves (1.17).

Similarly one obtains, using (1.23) and (1.20), which holds without assuming (1.14), that

$$\sum_{j=j_n}^{\infty} \{\varphi^*([\beta(n, j+1)])\}^{-1}$$
$$= \sum_{j=j_n}^{\infty} \{\varphi^*(\beta(n, j+1))/\varphi^*([\beta(n, j+1)])\}\{\varphi^*(\beta(n, j+1))\}^{-1}$$
$$\leqslant e^C \sum_{j=j_n}^{\infty} \{\varphi^*(\beta(n, j+1))\}^{-1} < e^C \int_{j_n}^{\infty} \{\varphi^*(\beta(n, x))\}^{-1}\, dx$$
$$= e^C \int_n^{\infty} \{g'(t)/\varphi^*(t)\}\, dt = -e^C \int_n^{\infty} \{1/\varphi^*(t)\}' \{g'(t)/g^{*\prime}(t)\}\, dt$$
$$< -e^C C \int_n^{\infty} \{1/\varphi^*(t)\}'\, dt = e^C C/\varphi^*(n),$$

for each $n \geqslant n_0$, and the proof is complete.

LEMMA 4. *Let $\varphi^*$, $\varphi$, $\psi \in \Omega$ with $\varphi^*(x) = e^{g^*(x)}$, $\psi(x) = e^{h(x)}$ and*

$$0 < \liminf_{x\to\infty} g^{*\prime}(x)/h'(x) < \infty. \tag{1.24}$$

*If there is $n_0 \in \mathbf{N}$ with $n_0 > \max(x_{\varphi^*}, x_\psi) + 1$ such that*

$$g^{*\prime}(x)/g'(x) \geqslant C > 1, \qquad x \geqslant n_0, \tag{1.25}$$

*and if $j_n = j_n(\psi, n_0)$, $\beta(n, x) = \beta_{\psi, n_0}(n, x)$ are associated to $\psi$ by (1.7), (1.8), then*

$$\sum_{j=0}^{j_n} \{\varphi^*([\beta(n, j+1)])/\varphi([\beta(n, j)])\}$$
$$= \mathcal{O}(\varphi^*(n)/\varphi(n)), \qquad n \to \infty, \tag{1.26}$$

$$\sum_{j=j_n}^{\infty} \{\varphi([\beta(n, j+1)])/\varphi^*([\beta(n, j)])\}$$
$$= \mathcal{O}(\varphi(n)/\varphi^*(n)), \qquad n \to \infty. \tag{1.27}$$

*Proof.* By (1.25) and Lemma 1(c) the function $\Phi(x) = \varphi^*(x)/\varphi(x)$ belongs to $\Omega$ and (cf. (1.1)) $x_\Phi + 1 < n_0$. Moreover, (1.25) implies

$$\limsup_{x\to\infty} g'(x)/g^{*\prime}(x) < 1$$

so that, using (1.24),

$$\liminf_{x\to\infty} (\log \Phi)'(x)/h'(x)$$
$$= \liminf_{x\to\infty} (g^{*\prime}(x)/h'(x))\{1 - (g'(x)/g^{*\prime}(x))\} > 0.$$

Thus Lemma 3(b) with $\varphi^*$, $\varphi$ replaced by $\Phi$, $\psi$, respectively, yields

$$\sum_{j=0}^{j_n} \Phi([\beta(n,j)]) = \mathcal{O}(\Phi(n)), \qquad n \to \infty,$$

$$\sum_{j=j_n}^{\infty} \{\Phi([\beta(n,j+1)])\}^{-1} = \mathcal{O}(1/\Phi(n)), \qquad n \to \infty,$$

where $\beta(n,x) = \beta_{\psi,n_0}(n,x)$, $j_n = j_n(\psi, n_0)$ (observing that $n_0 > \max(x_\psi, x_\Phi) + 1$), and this implies (1.26), (1.27) provided that $\varphi^*([\beta(n,j+1)])/\varphi^*([\beta(n,j)])$ is uniformly bounded with respect to $j \in \mathbf{P}$ and $n \geqslant n_0$. But this is a consequence of (1.24) and Lemma 3(a) with $\varphi$ replaced by $\psi$.

## 2. Zamansky-Type Theorem and Its Converse

We use the following notations (as in [5]). By $X$ we denote a normed linear space (NLS) and by $\{M_n\}_{n\in\mathbf{P}}$ a sequence of linear subspaces of $X$. Elements of $M_n$ are denoted by $p$, $p_n$; in particular $p_n^0 = p_n^0(f)$ stands for an element of best approximation to $f \in X$ from $M_n$, if it exists.

$$E_n[f] = \inf_{p\in M_n} \|f - p\|_x \qquad (= \|f - p_n^0\|_X)$$

denotes the error of best approximation. Moreover, $Y$ and $Z$ will be linear subspaces of $X$, equipped with seminorms $|\cdot|_Y$, $|\cdot|_Z$, respectively. The following basic assumptions will be made.

$$\lim_{n\to\infty} E_n[f] = 0, \tag{W}$$

$$\text{for each } f \in X,\ n \in \mathbf{P} \text{ there exists } p_n^0(f) \text{ with } \|f - p_n^0\| = E_n[f], \tag{E}$$

$$M_n \subset M_{n+1} \qquad \forall n \in \mathbf{P}, \tag{M}$$

$$M_n \subset Y \qquad \forall n \in \mathbf{P}. \tag{$\mathrm{S_Y}$}$$

Supposing that $X$, $Y$, $\{M_n\}_{n\in\mathbf{P}}$ satisfy $(\mathrm{S_Y})$, we say that $\{M_n\}$ satisfies a *Jackson-type inequality* of order $\varphi \in \Omega$ with respect to $Y$ if there exist constants $N$, $c = c(\varphi)$, independent of $n$, such that

$$E_n[f] \leqslant c(\varphi(n))^{-1} |f|_Y, \qquad f \in Y,\ n \geqslant N,\ n \in \mathbf{P}. \tag{$\mathrm{J_Y}$}$$

$\{M_n\}$ is said to satisfy a *Bernstein-type inequality* of order $\varphi \in \Omega$ with respect to $Y$ if there exist $N$, $c(\varphi)$ such that

$$|p_n| \leqslant c\varphi(n) \|p_n\|_X, \qquad p_n \in M_n,\ n \geqslant N,\ n \in \mathbf{P}. \tag{$\mathrm{B_Y}$}$$

The following two propositions are generalizations of Zamansky's theorem and its converse. The first steps of their proofs are similar to those in the proofs of [5, Lemmas 8 and 9], respectively.

PROPOSITION 1. *For a* NLS $X$ *and a subspace* $Y$, *let* $\{M_n\}_{n\in\mathbf{P}}$ *satisfy* (M), $(S_Y)$ *and* $(B_Y)$ *with an order* $\varphi^* \in \Omega$. *If, for some* $f \in X$, *there is a sequence* $\{p_n\}_{n\in\mathbf{P}}$ *with* $p_n \in M_n$ *for each* $n \in \mathbf{P}$ *and an order* $\varphi \in \Omega$ *such that* (1.5) *holds and*

$$\|f - p_n\|_X = \mathcal{O}(1/\varphi(n)), \qquad n \to \infty, \tag{2.1}$$

*then*

$$|p_n|_Y = \mathcal{O}(\varphi^*(n)/\varphi(n)), \qquad n \to \infty. \tag{2.2}$$

*Proof.* By (1.5) and (2.1) there exist numbers $C_1$, $C_2$, $n_1$, $n_2$ such that

$$g^{*\prime}(x)/g'(x) \geqslant C_1 > 1, \qquad x \geqslant n_1, \tag{2.3}$$

$$\|f - p_n\|_X \leqslant C_2/\varphi(n), \qquad n \geqslant n_2. \tag{2.4}$$

Let $n_0 \in \mathbf{N}$, $n_0 > \max(x_{\varphi^*}, x_\varphi, n_1, n_2, N) + 1$, where $N$ is given by $(B_Y)$, and let $j_n = j_n(\varphi^*, n_0)$, $\beta(n, x) = \beta_{\varphi^*, n_0}(n, x)$ be associated to $\varphi^*$ by (1.7), (1.8). By (1.7) and (1.1) there exists $n_3 \geqslant n_0$ such that $j_n \geqslant 1$ for each $n \geqslant n_3$. For such $n$ we have, using (1.11) and $(S_Y)$,

$$\begin{aligned} |p_n|_Y &= |p_{[\beta(n,j_n)]}|_Y \\ &\leqslant \sum_{j=0}^{j_n-1} |p_{[\beta(n,j+1)]} - p_{[\beta(n,j)]}|_Y + |p_{[\beta(n,0)]}|_Y. \end{aligned} \tag{2.5}$$

By (1.10), the last term is uniformly bounded in $n \geqslant n_0$, and, since $[\beta(n, j+1)] \geqslant [\beta(n, j)]$ by Lemma 2(b), the difference in the sum belongs to $M_{[\beta(n,j+1)]}$ in view of (M), so that $(B_Y)$ and (2.4) can be applied:

$$\begin{aligned} &|p_{[\beta(n,j+1)]} - p_{[\beta(n,j)]}|_Y \\ &\quad \leqslant c\varphi^*([\beta(n, j+1)])\, \|p_{[\beta(n,j+1)]} - p_{[\beta(n,j)]}\|_X \\ &\quad \leqslant c\varphi^*([\beta(n, j+1)])\{\|p_{[\beta(n,j+1)]} - f\|_X + \|p_{[\beta(n,j)]} - f\|_X\} \\ &\quad \leqslant 2cC_2\varphi^*([\beta(n, j+1)])/\varphi([\beta(n, j)]), \qquad 0 \leqslant j \leqslant j_n - 1. \end{aligned}$$

Hence (2.5) gives

$$|p_n| \leqslant 2cC_2 \sum_{j=0}^{j_n-1} \varphi^*([\beta(n, j+1)])/\varphi([\beta(n, j)]) + C, \qquad n \geqslant n_0.$$

Now, by (2.3), Lemma 4 can be applied with $\psi(x) = \varphi^*(x)$, so that (1.26) yields

$$|p_n|_Y = \mathcal{O}(\varphi^*(n)/\varphi(n)) + \mathcal{O}(1), \qquad n \to \infty.$$

Since $\Phi(x) = \varphi^*(x)/\varphi(x)$ is in $\Omega$, as has been shown in the proof of Lemma 4, $\Phi(x)$ tends to infinity as $x \to \infty$, and (2.2) follows.

PROPOSITION 2. *Let* $X$, $Y$, $\{M_n\}_{n \in \mathbf{P}}$ *satisfy* (W), (E), $(\mathrm{S_Y})$, *and* $(\mathrm{J_Y})$ *with an order* $\varphi^* \in \Omega$. *If, for some* $f \in X$, *there is an order* $\varphi \in \Omega$ *satisfying* (1.5) *and* (1.14) *and if there is a sequence* $\{p_n^0\}_{n \in \mathbf{P}}$ *of elements of best approximation to* $f$ *with* $p_n^0 \in M_n$ *for each* $n$ *such that*

$$|p_n^0|_Y = \mathcal{O}(\varphi^*(n)/\varphi(n)), \qquad n \to \infty, \tag{2.6}$$

*then*

$$E_n[f] = \mathcal{O}(1/\varphi(n)), \qquad n \to \infty. \tag{2.7}$$

*Proof.* Since $\Phi(x) = \varphi^*(x)/\varphi(x) \in \Omega$ by (1.5) and Lemma 1, there is some $x_\Phi \geqslant 0$ such that $\Phi$ satisfies the conditions (1.1) on the interval $[x_\Phi, \infty)$. By (1.14) we have

$$\begin{aligned}
&\liminf_{x \to \infty} g^{*\prime}(x)/(\log \Phi)'(x) \\
&\qquad = \liminf_{x \to \infty} \{1 - (g'(x)/g^{*\prime}(x))\}^{-1} \\
&\qquad = \{1 - (\limsup_{x \to \infty} g^{*\prime}(x)/g'(x))^{-1}\}^{-1} > 1,
\end{aligned}$$

thus there are constants $C$, $n_1$ such that

$$g^{*\prime}(x)/(\log \Phi)'(x) \geqslant C > 1, \qquad x \geqslant n_1. \tag{2.8}$$

Moreover, by (2.6), there are constants $C$, $n_2$ such that

$$|p_n^0|_Y \leqslant C\Phi(n), \qquad n \geqslant n_2. \tag{2.9}$$

Now let $n_0 \in \mathbf{N}$, $n_0 > \max(x_{\varphi^*}, x_\Phi, n_1, n_2, N) + 1$, where $N$ is given by $(\mathrm{J_Y})$, and let $j_n = j_n(\varphi^*, n_0)$, $\beta(n, x) = \beta_{\varphi^*, n_0}(n, x)$ be associated to $\varphi^*$ by (1.7), (1.8). Using (1.11), (E), $(\mathrm{S_Y})$, $(\mathrm{J_Y})$ and $E_n[f] \leqslant \|f\|_X$, we have

$$\begin{aligned}
E_n[f] = E_{[\beta(n,j_n)]}[f] &\leqslant E_{[\beta(n,j_n)]}[f - p^0_{[\beta(n,j_n+1)]}] \\
&\quad + E_{[\beta(n,j_n)]}[p^0_{[\beta(n,j_n+1)]}] \leqslant E_{[\beta(n,j+1)]}[f] \\
&\quad + c\varphi^*([\beta(n, j_n)])^{-1} |p^0_{[\beta(n,j_n+1)]}|_Y, \qquad n \geqslant n_0.
\end{aligned}$$

An $m$-fold iteration of this inequality gives

$$E_n[f] \leqslant E_{[\beta(n,j_n+m)]}[f] + c \sum_{j=j_n}^{j_n+m} \{\varphi^*([\beta(n,j)])\}^{-1} |p^0_{[\beta(n,j+1)]}|_Y,$$

and in view of (W) and (1.12), we may let $m \to \infty$ to obtain

$$E_n[f] \leqslant c \sum_{j=j_n}^{\infty} \{\varphi^*([\beta(n,j)])\}^{-1} |p^0_{[\beta(n,j+1)]}|_Y \tag{2.10}$$

or, after inserting (2.9),

$$E_n[f] \leqslant cC \sum_{j=j_n}^{\infty} \Phi([\beta(n,j+1)])/\varphi^*([\beta(n,j)]), \qquad n \geqslant n_0.$$

By (2.8), condition (1.25) of Lemma 4 is satisfied with $\psi = \varphi^*$ and $\varphi$ replaced by $\Phi$, so that (1.27) then yields the assertion.

Combining Proposition 2 with the particular case $p_n = p_n^0(f)$ of Proposition 1 we have

THEOREM 1. *Let $X$, $Y$, $\{M_n\}_{n\in\mathbf{P}}$ satisfy* (W), (E), (M), ($S_Y$), ($B_Y$) *and* ($J_Y$) *with an order $\varphi^* \in \Omega$, and let $\varphi \in \Omega$ satisfy* (1.5) *and* (1.14). *Then, for any $f \in X$,* (2.6) *and* (2.7) *are equivalent.*

## 3. FUNDAMENTAL THEOREM OF BEST APPROXIMATION

In the following theorem we use the $K$-functional with respect to a NLS $X$ and a subspace $Y \subset X$ with seminorm $|\cdot|_Y$, defined as usual by

$$K(t, f; X, Y) = \inf_{h\in Y} (\|f-h\|_X + t|h|_Y) \qquad (f \in X,\ t > 0). \tag{3.1}$$

THEOREM 2. *Let $X$, $Y$, $\{M_n\}$ satisfy* (W), (E), (M), ($S_Y$), ($B_Y$) *and* ($J_Y$) *with an order $\varphi^* \in \Omega$. Let $Z$ be another subspace of $X$ with seminorm $|\cdot|_Z$ such that $Z$ is a Banach space under the norm $\|\cdot\|_Z = \|\cdot\|_X + |\cdot|_Z$, and ($S_Z$), ($B_Z$), ($J_Z$) are satisfied with some order $\varphi_*(x) = \exp g_*(x) \in \Omega$. Then, for each $\varphi \in \Omega$ with the properties*

($\alpha$) $\liminf_{x\to\infty} g'(x)/g'_*(x) > 1$,

($\beta$) $\liminf_{x\to\infty} g^{*\prime}(x)/g'(x) > 1$,

($\gamma$) $\limsup_{x\to\infty} g^{*\prime}(x)/g'(x) < \infty$, *the following conditions are equivalent*:

(i) $E_n[f] = \mathcal{O}(1/\varphi(n))$, $n \to \infty$,

(ii) $|p_n^0|_Y = \mathcal{O}(\varphi^*(n)/\varphi(n))$, $n \to \infty$,

(iii) $f \in Z$ *and* $|f - p_n^0|_Z = \mathcal{O}(\varphi_*(n)/\varphi(n))$, $n \to \infty$,

(iv) $f \in Z$ *and* $E_n[f; z] = \mathcal{O}(\varphi_*(n)/\varphi(n))$, $n \to \infty$,

(v) $K(1/\varphi^*(x), f; X, Y) = \mathcal{O}(1/\varphi(x))$, $x \to \infty$.

*Here* $p_n^0$ *denotes an element of best approximation* (*cf.* (E)), *and* $E_n[f; Z]$ *stands for the error of best approximation with respect to the* $\|\cdot\|_Z$-*norm*: $E_n[f; Z] = \inf_{p \in M_n} \|f - p\|_Z$.

*Proof.* Noting that properties ($\beta$) and ($\gamma$) are identical with (1.5), (1.14), respectively, conditions (i) and (ii) are equivalent in view of Theorem 1, and the implication (v) $\Rightarrow$ (i) is an obvious consequence of (3.1).

(i) $\Rightarrow$ (v): By (3.1), ($S_Y$), the implication (i) $\Rightarrow$ (ii), and Lemma 1(b) we have, for $x$ large enough,

$$\begin{aligned}
&K(1/\varphi^*(x), f; X, Y) \\
&\quad \leqslant \|f - p_{[x]+1}^0\|_X + |p_{[x]+1}^0|_Y/\varphi^*(x) \\
&\quad \leqslant C\{1/\varphi([x]+1) + \varphi^*([x]+1)/(\varphi([x]+1)\,\varphi^*(x))\} \\
&\quad \leqslant (C/\varphi(x))\{1 + \varphi^*(x+1)/\varphi^*(x)\} \\
&\quad = \mathcal{O}(1/\varphi(n)), \qquad n \to \infty,
\end{aligned}$$

which implies (v).

(i) $\Rightarrow$ (iii): Let $n_0 \in \mathbf{N}$ be large enough to ensure that $n_0 > \max(x_\varphi, N) + 1$, where $N$ is given by ($B_Z$), and that (i) and ($\alpha$) may be written as

$$E_n[f] \leqslant C/\varphi(n), \qquad n \geqslant n_0, \tag{3.2}$$

$$g'(x)/g'_*(x) \geqslant C > 1, \qquad x \geqslant n_0, \tag{3.3}$$

respectively. Let $j_n = j_n(\varphi, n_0)$ and $\beta(n, x) = \beta_{\varphi, n_0}(n, x)$ be associated to $\varphi$ by (1.7), (1.8). In view of (1.11) and (3.2) one then has, for each $n \geqslant n_0$,

$$\begin{aligned}
&\sum_{j=j_n}^{\infty} \varphi_*([\beta(n, j+1)])\, E_{[\beta(n,j)]}[f] \\
&\quad \leqslant C \sum_{j=j_n}^{\infty} \varphi_*([\beta(n, j+1)])/\varphi([\beta(n, j)]),
\end{aligned}$$

and, if in Lemma 4 one replaces $\psi$ and $\varphi^*$ by $\varphi$ and $\varphi$ by $\varphi_*$, conditions (1.24), (1.25) are satisfied in view of (3.3), so that (1.27) yields

$$\sum_{j=j_n}^{\infty} \varphi_*([\beta(n, j+1)])\, E_{[\beta(n,j)]}[f] = \mathcal{O}(\varphi_*(n)/\varphi(n)), \qquad n \to \infty. \tag{3.4}$$

By (3.3) and Lemma 1(c), the function $\Phi(x) = \varphi(x)/\varphi_*(x)$ belongs to $\Omega$. In particular, $\Phi(x)$ tends to infinity for $x \to \infty$, and

$$\sum_{j=j_n}^{\infty} \varphi_*([\beta(n, j+1)])\, E_{[\beta(n,j)]}[f] = \mathcal{O}(1), \qquad n \to \infty. \tag{3.5}$$

Now an obvious modification of the proof of Lemma 7 in [5], (3.5) being the substitute for condition (11) there, yields that $f \in Z$ and

$$|f - p_n^0|_Z \leqslant \sum_{j=j_n}^{\infty} \varphi_*([\beta(n, j+1)])\, E_{[\beta(n,j)]}[f], \qquad n \geqslant n_0, \tag{3.6}$$

which implies $|f - p_n^0|_Z = \mathcal{O}(\varphi_*(n)/\varphi(n))$, $n \to \infty$, by (3.4). Since $\|f - p_n^0\|_X = \mathcal{O}(1/\varphi(n)) = o(\varphi_*(n)/\varphi(n))$, $n \to \infty$, assertion (iii) follows.

(iii) $\Rightarrow$ (iv): trivial.

(iv) $\Rightarrow$ (i): By $(\mathrm{S}_Y)$ and $(\mathrm{J}_Y)$ one has

$$\begin{aligned} E_n[f] = E_n[f - p_n] &\leqslant (c/\varphi_*(n))\, |f - p_n|_Z \\ &\leqslant (c/\varphi_*(n))\, \|f - p_n\|_Z, \end{aligned}$$

and taking the infimum over $p_n \in M_n$ for fixed $n$ and using (iv), $f \in Z$, $n \geqslant N$, $p_n \in M_n$, it follows that

$$E_n[f] \leqslant (c/\varphi_*(n))\, E_n[f, Z] = \mathcal{O}(1/\varphi(n)), \qquad n \to \infty,$$

which completes the proof.

## 4. Relation to Previous Results; Questions of Optimality

As already mentioned, Theorem 2 extends Theorems 1 and 2 of [5] by admitting a much larger set of orders (the convexity conditions for the class $\Phi$ of orders in [5] being relaxed and the lower bound for the rate of increase being removed in the definition of the new class $\Omega$). As is easily checked, our conditions $(\beta) + (\gamma)$ in Theorem 2 reduce to conditions (b) + (c) of Theorems 1 and 2 of [5], and our condition $(\alpha)$ is equivalent to (a) there, if $\varphi$, $\varphi^*$, $\varphi_*$ are restricted to the smaller class $\Phi$.

As in [5, p. 19], conditions ($\alpha$) and ($\beta$) require that the rate of growth of $\varphi$ lies between those of $\varphi^*$ and $\varphi_*$, and that there is a certain minimal distance between $\varphi$ and these two borders ($\varphi$ may now come closer to these borders than in [5]). It should be possible, however, to weaken ($\alpha$) and ($\beta$) still further. In fact, the original Zamansky inequality [11] did not require any distance between $\varphi$ and $\varphi^*$, and even allowed $\varphi$ to grow faster than $\varphi^*$. We will not pursue this aspect here. Our aim in this section is to investigate whether condition ($\gamma$) of Theorem 2 can be weakened and to study what may happen when ($\gamma$) is violated, thereby extending some of the results of [8, 9].

Condition ($\gamma$) requires that the orders $\varphi$ and $\varphi^*$ are not too far apart from each other (e.g., ($\gamma$) does admit the pair $\varphi^*(x) = x$, $\varphi(x) = x^\alpha$, $\alpha > 0$ but not $\varphi(x) = \log x$), and it guarantees that the order $\varphi^*(n)/\varphi(n)$ in the implication (i) $\Rightarrow$ (ii) of Theorem 2 is sharp. Indeed, if the hypotheses of Theorem 2 are satisfied and if $f \in X$ is such that (i) holds and cannot be improved, i.e., $E_n[f] = \mathcal{O}(1/\varphi(n))$, $n \to \infty$, and $1/\varphi(n) = \mathcal{O}(E_n[f])$, $n \to \infty$, which we abbreviate to

$$E_n[f] \approx 1/\varphi(n), \qquad n \to \infty, \tag{4.1}$$

and if the corresponding $p_n^0$ would satisfy a better estimate than (ii), i.e.,

$$|p_n^0|_Y = \mathcal{O}(\rho(n)), \quad n \to \infty, \qquad \rho(n) = o(\varphi^*(n)/\varphi(n)), \quad n \to \infty, \tag{4.2}$$

then, even if $\rho$ does not belong to our grid of orders $\Omega$, as $\varphi^*/\varphi$ does, the proof of Proposition 2 can be repeated with (2.6) replaced by (4.2) to arrive at $E_n[f] = o(1/\varphi(n))$, $n \to \infty$, which contradicts (4.1). Thus ($\gamma$) is a sufficient condition for (4.1) to imply

$$|p_n^0|_Y \approx \varphi^*(n)/\varphi(n), \qquad n \to \infty, \tag{4.3}$$

where we assume, of course, that also the other hypotheses of Theorem 2 are satisfied.

A partial converse of this implication will be obtained as a corollary to the next theorem, namely the statement that, for a certain subset of the set of those pairs $(\varphi^*, \varphi)$ which satisfy ($\beta$), there are particular spaces $X$, $Y$, and a sequence $\{M_n\}$ such that from the validity of the implication (4.1) $\Rightarrow$ (4.3) for such a pair one can conclude the validity of ($\gamma$). The subset there will consist of pairs $(\varphi^*, \varphi)$ for which, apart from some restriction upon $\varphi^*$, the function $\varepsilon(x) = g^{*\prime}(x)/g'(x)$ exists, has a positive and continuous derivative, and satisfies condition (4.4) below. As is shown in the following lemma, this condition serves to prevent the slope of $\varepsilon(x)$ from oscillating too much (cf. (4.6)), and it imposes an upper bound to the rate of increase of $\varepsilon(x)$ (cf. (4.5))). The latter restriction is not essential since, in view of the boundedness of $\varepsilon(x)$ required by condition ($\gamma$), we are just interested in pairs $\varphi^*$, $\varphi$ for which $\varepsilon(x)$ increases rather slowly (cf. also Remark 3 below).

LEMMA 5. *Let $\varepsilon(x)$ be a positive, strictly increasing function on $(x_0, \infty)$ for some $x_0 \geqslant 0$ with continuous derivative, such that*

$$A = \limsup_{x\to\infty} x\varepsilon'(x) < c_1 \tag{4.4}$$

*for some constant $c_1 > 0$. Then*

(i) *there are constants $B \in (A, c_1)$ and $x_1 > x_0$ such that*

$$\varepsilon(x) \leqslant B \log x, \qquad x \geqslant x_1; \tag{4.5}$$

(ii) *for each $a > 1$ one has*

$$\varepsilon(ax) = \mathcal{O}(\varepsilon(x)), \qquad x \to \infty; \tag{4.6}$$

(iii) *if, in addition,*

$$\liminf_{x\to\infty} \varepsilon(x) > 3, \tag{4.7}$$

*and if $\varphi^*(x) = e^{g^*(x)} \in \Omega$ satisfies*

$$0 < c_1 \leqslant xg^{*\prime}(x), \qquad x \geqslant x_{\varphi^*}, \tag{4.8}$$

*for some constant $x_{\varphi^*}$, then there exists an $x_2 > \max(x_0, x_{\varphi^*})$ such that*

$$\{g^{*\prime}(x)(\varepsilon(x) - 1)\}^{-1} \leqslant \{2\varepsilon'(x)\}^{-1}, \qquad x \geqslant x_2. \tag{4.9}$$

*Proof.* (i) By (4.4) there are numbers $B' \in (A, c_1)$ and $x_1' > x_0$ such that $x\varepsilon'(x) < B'$ for each $x \geqslant x_1'$, whence (4.5) follows in view of

$$\begin{aligned}\varepsilon(x) &= \int_{x_1'}^{x} \varepsilon'(t)\, dt + \varepsilon(x_1') < B'(\log x - \log x_1') + \varepsilon(x_1') \\ &< B \log x, \qquad x > x_1',\end{aligned}$$

where $B \in (B', c_1)$ and $x_1 > x_1'$ are suitably chosen.

(ii) Applying the mean value theorem to $\log \varepsilon(x)$ and using the monotonicity of $\varepsilon(x)$ as well as (4.4), there is a $\xi = \xi(x) \in (x, ax)$ such that, for large $x$,

$$\begin{aligned}\log \frac{\varepsilon(ax)}{\varepsilon(x)} &= (a-1)\, x(\log \varepsilon)'(\xi) = (a-1)\, x\varepsilon'(\xi)/\varepsilon(\xi) \\ &< (a-1)\, x\varepsilon'(\xi)/\varepsilon(x_0+1) < (a-1)\, xc_1(\varepsilon(x_0+1)\xi)^{-1} \\ &< c_1(a-1)/\varepsilon(x_0+1),\end{aligned}$$

which implies (4.6).

(iii) Choosing $x_2 > \max(x_0, x_{\varphi^*})$ large enough such that $x \geqslant x_2$ implies $x\varepsilon'(x) \leqslant c_1$ (cf. (4.4)) as well as $\varepsilon(x) \geqslant 3$ (cf. (4.7)), it follows by (4.8) that

$$\varepsilon'(x) \leqslant c_1/x \leqslant g^{*\prime}(x) \leqslant \tfrac{1}{2} g^{*\prime}(x)(\varepsilon(x) - 1), \qquad x \geqslant x_2,$$

and this gives (4.9) since $\varepsilon'(x)$ is positive.

In the following theorem, $C_{2\pi}$ denotes the space of continuous, $2\pi$-periodic functions, and $\Pi_n$ stands for the set of trigonometric polynomials of degree $\leqslant n$.

THEOREM 3. *Let $X = C_{2\pi}$, $M_n = \Pi_n$ for each $n \in \mathbf{P}$, and let $Y \subset C_{2\pi}$ be such that* $(\mathrm{S}_Y)$ *and* $(\mathrm{B}_Y)$ *are satisfied with an order $\varphi^*(x) = e^{g^*(x)} \in \Omega$ for which $xg^{*\prime}(x)$ is non-increasing on $[x_{\varphi^*}, \infty)$ and for which* (4.8) *holds with an $x_{\varphi^*} > 1$. Given any function $\varepsilon(x)$ with the properties as in Lemma* 5 *(including* (4.7), *and with the $c_1$ in* (4.4) *being the same as that in* (4.8)), *then there exist a $\varphi \in \Omega$ and an $x^* > x_{\varphi^*}$ such that conditions ($\beta$) of Theorem* 2 *as well as*

$$g^{*\prime}(x)/g'(x) = \varepsilon(x), \qquad x \geqslant x^*, \tag{4.10}$$

*are satisfied, and there is an $f \in C_{2\pi}$ such that, in the notation of* (4.1),

$$E_n[f] \approx 1/\varphi(n), \qquad n \to \infty, \tag{4.11}$$

$$|p_n^0(f)|_Y = \mathcal{O}\left(\frac{1}{\varepsilon(n)}\frac{\varphi^*(n)}{\varphi(n)}\right), \qquad n \to \infty. \tag{4.12}$$

*Proof.* $x^* > \max(x_{\varphi^*}, x_1, x_2)$ be fixed, where $x_1$, $x_2$ are given by (4.5), (4.9), respectively. Since $\varepsilon(x)$ and $g^{*\prime}(x)$ are continuous and positive for $x \geqslant x^*$, the function

$$g(x) = \int_{x^*}^{x} g^{*\prime}(t)/\varepsilon(t)\, dt \tag{4.13}$$

is well defined for each $x \geqslant x^*$, satisfies (4.10) as well as $g'(x) > 0$ for each $x > x^*$, and, by (4.8), (4.5), one has

$$\begin{aligned} g(x) &\geqslant c_1 \int_{x^*}^{x} (t\varepsilon(t))^{-1}\, dt \\ &\geqslant (c_1/B) \int_{x^*}^{x} (t \log t)^{-1}\, dt \to \infty, \qquad x \to \infty. \end{aligned}$$

Hence the function $\varphi(x) = e^{g(x)}$ belongs to $\Omega$ (cf. (1.1)), with $x_\varphi = x^* + 1$, say, and condition ($\beta$) of Theorem 2 is satisfied in view of (4.7), (4.10). By

Lemma 1(c), this also implies that $\varphi^*(c)/\varphi(x) \in \Omega$, and hence there is some $x_3 \geqslant x_\varphi$ such that

$$g^{*\prime}(x) - g'(x) > 0, \qquad x \geqslant x_3. \tag{4.14}$$

We want to show that (4.11) and (4.12) hold for the following function $f$:

$$f(x) = \sum_{k=0}^{\infty} a(k) \cos 3^k x, \tag{4.15}$$

where, for some $n_0 \geqslant x_3$, we set

$$\begin{aligned} a(x) &= 1; && 0 \leqslant x < n_0, \\ &= (\varphi(3^x))^{-1} g'(3^x)\, 3^x \log 3; && x \geqslant n_0. \end{aligned} \tag{4.16}$$

This $f$ is in $C_{2\pi}$ since (4.15) converges absolutely and uniformly for all $x$. Indeed, since $xg^{*\prime}(x)$ is non-increasing on $[x_{\varphi^*}, \infty)$, one has

$$xg^{*\prime}(x) \leqslant c_2, \qquad x \geqslant x_{\varphi^*}, \tag{4.17}$$

for some constant $c_2 \geqslant c_1$, which, together with (4.7), (4.10), (4.8), and (4.5), implies that

$$\begin{aligned} c_2(c_1 - B)(2c_1 x)^{-1} &\geqslant (c_1 - B)\, g^{*\prime}(x)/(2c_1) \\ &\geqslant g^{*\prime}(x)/\varepsilon(x) = g'(x) \geqslant c_1(x\varepsilon(x))^{-1} \\ &\geqslant (c_1/B)(x \log x)^{-1} \end{aligned} \tag{4.18}$$

for $x$ large enough, thus $g(x) \geqslant (c_1/B) \log\log x + C$, where $c_1/B > 1$ and $C$ is some further constant, and therefore

$$0 < a(k) = \mathcal{O}(\exp\{-(c_1/B) \log\log 3^k\}) = \mathcal{O}(k^{-c_1/B}), \qquad k \to \infty.$$

By Bernstein's theorem on lacunary Fourier series (see [10, p. 77]), for each $n \in \mathbf{N}$, the polynomial $p_n^0 \in \Pi_n$ of best approximation to $f$ is given by

$$p_n^0(f; x) = \sum_{k=0}^{v} a(k) \cos 3^k x, \tag{4.19}$$

where $v = v(n)$ is associated to $n$ by

$$3^v \leqslant n < 3^{v+1}, \tag{4.20}$$

and the error of best approximation is

$$E_n[f] = \sum_{k=v+1}^{\infty} a(k), \qquad n \in \mathbf{N}. \tag{4.21}$$

By (4.10) and the monotonicity assumptions on $xg^{*\prime}(x)$ and $\varepsilon(x)$, both $xg'(x)$ and $a(x)$ are decreasing, so that, for $n \geqslant 3^{n_0}$, say, (4.21) and (4.16) yield

$$1/\varphi(3^{\nu+1}) = \int_{\nu+1}^{\infty} a(x)\,dx \leqslant \sum_{k=\nu+1}^{\infty} a(k) = E_n[f]$$
$$\leqslant \int_{\nu}^{\infty} a(x)\,dx = 1/\varphi(3^{\nu}),$$

or, in view of (4.20),

$$1/\varphi(3n) \leqslant E_n[f] < 1/\varphi(n/3), \qquad n \geqslant 3^{n_0}. \tag{4.22}$$

Now, replacing the $c_1$ in (4.4) by $c_2$, for example, Lemma 5(ii) may be applied to $\varphi(x)$ instead of $\varepsilon(x)$ (cf. the left-hand side of (4.18)), which gives

$$\varphi(3x) \approx \varphi(x), \qquad x \to \infty, \tag{4.23}$$

and so (4.11) follows from (4.22).

For the proof of (4.12) we use (4.19), $(\mathrm{S}_\mathrm{Y})$, and $(\mathrm{B}_\mathrm{Y})$, which imply

$$|p_n^0|_Y \leqslant \sum_{k=0}^{\nu} a(k)\,|\cos 3^k x|_Y \leqslant C \sum_{k=0}^{\nu} a(k)\,\varphi^*(3^k)$$
$$= C \sum_{k=0}^{\nu} \Phi(3^k), \tag{4.24}$$

where we have set (cf. (4.16))

$$\Phi(x) = 1; \qquad 1 \leqslant 3^x < n_0,$$
$$= (\varphi^*(x)/\varphi(x))\,g'(x)x; \qquad 3^x \geqslant n_0.$$

We split up the sum in (4.24) into two parts:

$$|p_n^0|_Y \leqslant C \left\{ \sum_{k=0}^{n_0-1} \Phi(3^k) + \sum_{k=n_0}^{\nu} \Phi(3^k) \right\}, \tag{4.25}$$

the first of which being constant. To estimate the second one we use (4.6), the inequality

$$c_1 \frac{\varphi^*(x)}{\varphi(x)} \frac{1}{\varepsilon(x)} \leqslant \frac{\varphi^*(x)}{\varphi(x)} \frac{xg^{*\prime}(x)}{\varepsilon(x)}$$
$$= \Phi(x) \leqslant c_2 \frac{\varphi^*(x)}{\varphi(x)} \frac{1}{\varepsilon(x)}, \qquad 3^x \geqslant n_0,$$

which follows by (4.10), (4.8), (4.17), and the fact that $\varepsilon(x)$ and $\varphi^*(x)/\varphi(x) \in \Omega$ are increasing functions for $x \geqslant n_0$ (cf. (4.14)), so that, for each $x \in [3^k, 3^{k+1}]$, $k \geqslant n_0$, one has

$$\begin{aligned}\Phi(3^k) &\leqslant \frac{\varphi^*(3^k)}{\varphi(3^k)} \frac{c_2}{\varepsilon(3^k)} \leqslant \frac{\varphi^*(x)}{\varphi(x)} \frac{c_2}{\varepsilon(3^k)} \leqslant C \frac{\varphi^*(x)}{\varphi(x)} \frac{1}{\varepsilon(3^{k+1})} \\ &\leqslant C \frac{\varphi^*(x)}{\varphi(x)} \frac{1}{\varepsilon(x)} \leqslant \frac{C}{c_1} \Phi(x). \end{aligned} \tag{4.26}$$

Hence it follows that

$$\begin{aligned}\sum_{k=n_0}^{\nu} \Phi(3^k) &= \sum_{k=n_0}^{\nu} (3^{k+1} - 3^k)^{-1} \Phi(3^k) \int_{3^k}^{3^{k+1}} dx \\ &= \frac{3}{2} \sum_{k=n_0}^{\nu} 3^{-k-1} \Phi(3^k) \int_{3^k}^{3^{k+1}} dx \\ &\leqslant C \sum_{k=n_0}^{\nu} 3^{-k-1} \int_{3^k}^{3^{k+1}} \Phi(x)\, dx \\ &\leqslant \sum_{k=n_0}^{\nu} \int_{3^k}^{3^{k+1}} (\Phi(x)/x)\, dx = C \int_{3^{n_0}}^{3^{\nu+1}} (\Phi(x)/x)\, dx. \end{aligned} \tag{4.27}$$

To evaluate the latter integral we use the representation

$$\Phi(x)/x = (\varphi^*/\varphi)'(x)(\varepsilon(x) - 1)^{-1}, \qquad x \geqslant n_0, \tag{4.28}$$

which is an immediate consequence of the definition of $\Phi$, $\varphi^*$, and $\varphi$, and of (4.10), (4.14). Setting $h(x) = \varphi^*(x)/\varphi(x)$, (4.10) and (4.9) imply

$$\begin{aligned} h(x)/h'(x) &= (g^{*\prime}(x) - g'(x))^{-1} = \varepsilon(x)\{g^{*\prime}(x)(\varepsilon(x) - 1)\}^{-1} \\ &\leqslant \varepsilon(x)\{2\varepsilon'(x)\}^{-1}, \end{aligned}$$

for each $x > \max(x_2, x^*)$, or

$$\frac{h'(x)}{\varepsilon(x)} = \left(\frac{h(x)}{\varepsilon(x)}\right)' + \frac{h(x)\,\varepsilon'(x)}{(\varepsilon(x))^2} \leqslant \left(\frac{h(x)}{\varepsilon(x)}\right)' + \frac{h'(x)}{2\varepsilon(x)}.$$

Inserting this into (4.28) we have

$$\begin{aligned} \Phi(x)/x &= h'(x)(\varepsilon(x) - 1)^{-1} \\ &\leqslant 2\varepsilon(x)\{\varepsilon(x) - 1\}^{-1}(h(x)/\varepsilon(x))', \qquad x > \max(x_2, n_0) \end{aligned}$$

which, together with (4.27), (4.6), and the monotonicity of $\varepsilon(x)$, gives

$$\sum_{k=n_0}^{\nu} \Phi(3^k) \leqslant C \int_{3^{n_0}}^{3^{\nu+1}} (\Phi(x)/x)\,dx \leqslant 2C \int_{3^{n_0}}^{3^{\nu+1}} \frac{\varepsilon(x)}{\varepsilon(x)-1} \left(\frac{h(x)}{\varepsilon(x)}\right)' dx$$

$$\leqslant 2C \frac{\varepsilon(3^{n_0})}{\varepsilon(3^{n_0}))-1} \int_{3^{n_0}}^{3^{\nu+1}} \left(\frac{h(x)}{\varepsilon(x)}\right)' dx$$

$$= C \frac{\varphi^*(3^{\nu+1})}{\varphi(3^{\nu+1})} \frac{1}{\varepsilon(3^{\nu+1})} + C, \qquad \nu \geqslant n_0.$$

Using the monotonicity of $\varphi$ and $\varepsilon$ again as well as the fact that $\varphi^*(ax) \approx \varphi^*(x)$ as $x \to \infty$ for any $a > 1$ (cf. (4.23) and its proof), we obtain

$$\sum_{k=n_0}^{\nu} \Phi(3^k) \leqslant C\varphi^*(3^\nu)\{\varphi(3^\nu)\,\varepsilon(3^\nu)\}^{-1} + C, \qquad \nu \geqslant n_0.$$

Now $\nu$ and $n$ are related by (4.20), so that (4.26) applies with $k$, $x$ replaced by $\nu$, $n$, respectively, giving

$$\sum_{k=n_0}^{\nu} \Phi(3^k) \leqslant C\varphi^*(n)\{\varphi(n)\,\varepsilon(n)\}^{-1} + C, \qquad n \geqslant 3^{n_0}.$$

In view of (4.25), assertion (4.12) will follow provided that

$$\lim_{n\to\infty} \varphi^*(n)\{\varphi(n)\,\varepsilon(n)\}^{-1} = +\infty. \tag{4.29}$$

To show this we use (4.13), the monotonicity of $\varepsilon(x)$, and (4.8):

$$\begin{aligned}
&\varphi^*(x)\{\varphi(x)\,\varepsilon(x)\}^{-1} \\
&\quad = \exp\{g^*(x) - g(x) - \log \varepsilon(x)\} \\
&\quad = \exp\left\{g^*(x) - \int_{x^*}^{x} g^{*\prime}(t)/\varepsilon(t)\,dt - \log \varepsilon(x)\right\} \\
&\quad = \exp\left\{\int_{x^*}^{x} g^{*\prime}(t)(1 - 1/\varepsilon(t))\,dt + g^*(x^*) - \log \varepsilon(x)\right\} \\
&\quad \geqslant \exp\left\{(1 - 1/\varepsilon(x^*))\,c_1 \int_{x^*}^{x} t^{-1}\,dt + g^*(x^*) - \log \varepsilon(x)\right\} \\
&\quad \geqslant \exp\{C + C\log x - \log \varepsilon(x)\}
\end{aligned}$$

for each $x > x^*$. In view of (4.5), the latter expression tends to $\infty$ as $x \to \infty$, so that (4.29) follows.

COROLLARY. *Let $X = C_{2\pi}$, $M_n = \Pi_n$ for each $n \in \mathbf{P}$ and denote by $\mathfrak{S}$ the set of pairs $(\varphi^*, \varphi)$ of orders $\varphi^*, \varphi \in \Omega$ with the following properties:*

(i) *$xg^{*\prime}(x)$ is non-increasing on $[x_{\varphi^*}, \infty)$ and satisfies (4.8) for some $x_{\varphi^*} > 1$,*

(ii) *there exists a subspace $Y \subset C_{2\pi}$ such that $(\mathrm{S_Y})$ and $(\mathrm{B_Y})$ hold with order $\varphi^*$,*

(iii) *the function*

$$\varepsilon(x) = g^{*\prime}(x)/g'(x), \tag{4.30}$$

*defined for all $x > \max(x_{\varphi^*}, x_\varphi)$, satisfies the properties required Lemma 5. If, for some pair $(\varphi^*, \varphi) \in \mathfrak{S}$, the implication (4.1) $\Rightarrow$ (4.3) is true, it follows that this pair satisfies condition $(\gamma)$ of Theorem 2.*

*Proof.* Let $(\varphi^*, \varphi) \in \mathfrak{S}$ be such that the implication (4.1) $\Rightarrow$ (4.3) is valid and assume that $(\gamma)$ fails to hold, i.e., in view of the monotonicity of $\varepsilon(x)$, $\lim_{x\to\infty} \varepsilon(x) = +\infty$. Then all the hypotheses on $\varphi^*(x)$ and $\varepsilon(x)$ in Theorem 3 are satisfied and, writing now $\psi$ instead of $\varphi$ there, Theorem 3 yields an $x^* > 1$, a $\psi(x) = \exp\{\int_{x^*}^x g^{*\prime}(t)/\varepsilon(t)\,dt\}$ and an $f \in C_{2\pi}$ for which $E_n[f] \approx 1/\psi(n)$, $|p_n^0(f)|_Y = \mathcal{O}(\varphi^*(n)(\varepsilon(n)\,\psi(n))^{-1})$, $n \to \infty$. By (4.30), $\psi(x) = \exp\{\int_{x^*}^x g'(t)\,dt\} = C\varphi(x)$ for each $x > x^*$, so that we have as well $E_n[f] \approx 1/\varphi(n)$, which is (4.1), and $|p_n^0(f)|_Y = \mathcal{O}(\varphi^*(n)(\varepsilon(n)\,\varphi(n))^{-1})$, $n \to \infty$. Since $1/\varepsilon(n)$ tends to zero as $n \to \infty$, the latter result contradicts (4.3), so that $(\gamma)$ must be satisfied.

*Remarks.* 1. Condition (ii) of the Corollary may be replaced by the assumption that $\varphi^* \in \Omega$ is defined on $[0, \infty)$ (thus $x_{\varphi^*} = 0$ in (1.1)) with $\varphi^*(0) = 0$ and that $\varphi^*(x)$ is concave or convex, which is the case, e.g., for $\varphi^*(x) = x^\alpha$, $\alpha > 0$. Indeed, for $Y$ one may then take

$$\begin{aligned} Y_{\varphi^*} = \{f \in C_{2\pi};\, \exists h \in C_{2\pi} \text{ such that } \varphi^*(|k|)\, f^\wedge(k) \\ = h^\wedge(k)\ \forall k \in \mathbf{Z}\}, \end{aligned} \tag{4.31}$$

where $f^\wedge(k) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\,dx$ denotes the $k$th Fourier coefficient of $f$, and $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Then $|f|_Y = \|h\|_{C_{2\pi}}$ defines a seminorm on $Y$ and properties $(\mathrm{S_Y})$, $(\mathrm{B_Y})$ are satisfied (for $(\mathrm{B_Y})$ cf., e.g., [7]; see also [1] for the particular case $\varphi^*(x) = x^\alpha$, $\alpha > 0$, where $Y$ is characterized via fractional derivatives and differences).

2. Returning to the discussion of Theorem 3, we note that its hypotheses admit functions $\varepsilon(x)$ which tend to infinity with an arbitrarily slow rate of increase. For example, one may take $\varphi^*(x) = x^\alpha$, $\alpha > 0$ with the corresponding $Y$ as defined by (4.31), and

$$\varepsilon(x) = l_r(x) \left\{ 1 - \left( \prod_{j=2}^{r} l_j(x) \right)^{-1} \right\}^{-1} \tag{4.32}$$

for any $r \in \mathbf{N}$, $r \geqslant 2$, where $l_r$ is defined by (1.2). Theorem 3 then yields (4.11), (4.12) with

$$\varphi(x) = x^{\alpha/l_r(x)}.$$

(The above choice for $\varepsilon(x)$ is preferrable to choosing $\varepsilon(x) = l_r(x)$ since (4.13) would then be an elliptic integral).

**3.** In the above example, the improved Zamansky type estimate (4.12) obtained is, of course, only slightly better than the original one (cf. (ii) in Theorem 2). The improving factor $\varepsilon(n)$ can, however, be even larger than $\varphi(n)$. Indeed, in Theorem 3 the restriction on the rate of growth of $\varepsilon(n)$ (see (4.5)) has been made for technical reasons only. Proceeding just as in the proof of Theorem 3 one can easily show that, with $\varphi^*$ and $Y$ chosen as in Remark 2, also

$$\varepsilon(x) = (\alpha/\rho) \log x, \qquad \rho > 0, \tag{4.33}$$

$$\varepsilon(x) = \alpha \prod_{j=1}^{s} l_j(x), \qquad s \in \mathbf{N}, \tag{4.34}$$

lead to similar results for each $\alpha > 0$. The corresponding $\varphi$'s are

$$\varphi(x) = (\log x)^\rho, \tag{4.35}$$

$$\varphi(x) = l_s(x), \tag{4.36}$$

respectively. For example, in case (4.34), (4.36) one obtains, if $s \geqslant 2$, that the function

$$f(x) = \sum_{k=0}^{\infty} b(k+c) \cos 3^k x,$$

where $c$ is a constant to be chosen suitably large, and

$$b(x) = \left\{ x(l_{s-1}(x))^2 \sum_{j=1}^{s-1} l_j(x) \right\}^{-1},$$

belongs to $C_{2\pi}$ and satisfies

$$E_n[f] \approx (l_s(n))^{-1}, \quad |p_n^0|_Y = \mathcal{O}\left(n^\alpha \left(l_s(n) \prod_{j=1}^{s} l_j(n)\right)^{-1}\right), \quad n \to \infty.$$

**4.** Comparing these examples with those given in [8] for exponential orders, we note that in [8] it appeared that the existence of an improved Zamansky-type inequality seems to be connected with a certain regular behaviour of the error of best approximation (at least for the spaces $X = L^2_{2\pi}$, see [8, Theorem 2] and $X = L^1_{2\pi}$, see [9]), whereas the functions whose Fourier series have "Bernstein gaps," thus for which the sequence $\{E_n[f]\}_{n \in \mathbf{P}}$ is piecewise constant, seemed to be the typical examples for which no improvement of Zamansky's inequality is possible, even in cases where the inverse Zamansky-type theorem fails to hold. But, as Theorem 3 shows, the situation is in fact not as simple, since there just such Bernstein gaps were employed to obtain an improvement. Moreover, as the next remark shows, gap series with broader gaps than those in Theorem 3 can be used to construct examples where $(\gamma)$ does not hold and nevertheless no improvement is possible.

5. Two further questions remain to be studied. Firstly the question whether (4.12) is best possible for general $f$ which satisfy (4.11). A positive answer to this will be given in Theorem 4 below. Secondly, it might be conjectured that, given a pair of orders $(\varphi^*, \varphi) \in \mathfrak{S}$ (cf. Corollary), an improved Zamansky-type estimate holds for *all* $f$ which satisfy (4.11). However, this is not the case, as the following example shows. Let $X = C_{2\pi}$, $M_n = \Pi_n$ $\forall n \in \mathbf{P}$, $\varphi^*(x) = x^\alpha$ for some $\alpha > 0$, and $\varphi(x) = \log x$, thus $(\gamma)$ is not satisfied. Defining the space $Y = Y_{\varphi^*}$ by (4.31) and a sequence $\{n_k\}_{k \in \mathbf{P}}$ by $n_0 = 2$, $n_{k+1} = (2p_k + 1)\, n_k$, where $p_k = 2^{2^{k-1}}$ for each $k \in \mathbf{P}$, the function $f$ defined by

$$f(x) = \sum_{k=0}^{\infty} a(k) \cos n_k x, \qquad a(x) = 2 \cdot 2^{-x},$$

belongs to $C_{2\pi}$ and satisfies the assumptions of Bernstein's theorem (cf. (4.19)–(4.21)). Observing that $2^{2^k} \leqslant n_k < 2^{2^{k+1}}$ for each $k \in \mathbf{P}$, one readily shows that

$$E_n[f] \approx 1/\varphi(n), \qquad n \to \infty,$$

and

$$2 \log 2 < \limsup_{n \to \infty} |p_n^0(f)|_Y \{\varphi^*(n)/\varphi(n)\}^{-1} < \infty.$$

(The mere existence of a function $f$ with the latter two properties may also be established as in [8, Theorem 1]. Though the Lemma used there is no more applicable here, a sequence of the desired shape can easily be constructed explicitly).

THEOREM 4. *Let* $X$, $Y$, $\{M_n\}_{n \in \mathbf{P}}$ *satisfy conditions* (W), (E), (M), $(\mathrm{S}_\mathrm{Y})$ *and* $(\mathrm{J}_\mathrm{Y})$ *with an order* $\varphi^*(x) = e^{g^*(x)} \in \Omega$, *for which*

$$g^{*\prime}(x+1)/g^{*\prime}(x) = \mathcal{O}(1), \qquad x \to \infty, \tag{4.37}$$

*and let* $\varphi(x) = e^{g(x)} \in \Omega$ *be such that, for some* $x_1 > \max(x_{\varphi^*}, x_\varphi)$,

$$g'(x)\{\varphi(x)\, g^{*\prime}(x)\}^{-1} \text{ is non-increasing for } x > x_1. \tag{4.38}$$

*If* $f \in X$ *satisfies*

$$\limsup_{n\to\infty} \varphi(n)\, E_n[f] > 0, \tag{4.39}$$

*then for each non-negative function* $\psi$ *with the property that*

$$|p_n^0(f)|_Y = \mathcal{O}(\psi(n)), \qquad n \to \infty, \tag{4.40}$$

*it follows that*

$$\limsup_{n\to\infty} \psi(n)\, \varphi(n)\, g^{*\prime}(n)\{\varphi^*(n)\, g'(n)\}^{-1} > 0. \tag{4.41}$$

*Proof.* In the proof of Proposition 2 it has been shown (see (2.10)) that (W), (E), ($S_Y$) and ($J_Y$) imply

$$E_n[f] \leqslant c \sum_{j=j_n}^{\infty} \varphi^*([\beta(n, j)])^{-1}\, |p^0_{[\beta(n,j+1)]}|_Y, \qquad n \geqslant n_0,$$

for some $n_0 \in \mathbf{N}$, where $j_n = j_n(\varphi^*, n_0)$ and $\beta(n, x) = \beta_{\varphi^*, n_0}(n, x)$ are associated to $\varphi^*$ by (1.7), (1.8), respectively. Assuming that there is a $\psi$ satisfying (4.40) but violating (4.41), it follows that

$$\begin{aligned} E_n[f] = {} & o\Bigg(\sum_{j=j_n}^{\infty} \frac{\varphi^*([\beta(n, j+1)])}{\varphi^*([\beta(n, j)])} \\ & \times \frac{g'([\beta(n, j+1)])}{\varphi([\beta(n, j+1)])\, g^{*\prime}([\beta(n, j+1)])}\Bigg), \qquad n \to \infty. \end{aligned}$$

In view of Lemma 3(a) (with $\varphi^* = \varphi$), the first quotient in the sum is uniformly bounded in $j$ and $n$, so that (4.38) implies

$$\begin{aligned} E_n[f] &= o\Bigg(\sum_{j=j_n}^{\infty} \Phi(\beta(n, j+1) - 1)\Bigg) \\ &= o\Bigg(\sum_{j=j_n+1}^{\infty} \Phi(\beta(n, j) - 1)\Bigg), \qquad n \to \infty, \end{aligned}$$

where we have set $\Phi(x) = g'(x)\{\varphi(x)\, g^{*\prime}(x)\}^{-1}$. The last sum can be majorized by an integral since $\Phi(\beta(n, x) - 1)$ is a decreasing function of $x$ in view of (4.38) and Lemma 2(b). Substituting $t = \beta(n, x) - 1$ and using (1.9) (where $g'$ has to be replaced by $g^{*\prime}$), (4.37), (1.11), (1.12), and (1.4), one obtains

$$\begin{aligned} E_n[f] &= o\left(\int_{j_n}^{\infty} \Phi(\beta(n,x)-1)\,dx\right) = o\left(\int_{n-1}^{\infty} \Phi(t)\, g^{*\prime}(t+1)\,dt\right) \\ &= o\left(\int_{n-1}^{\infty} \{g'(t)/\varphi(t)\}\{g^{*\prime}(t+1)/g^{*\prime}(t)\}\,dt\right) \\ &= o\left(\int_{n-1}^{\infty} \{g'(t)/\varphi(t)\}\,dt\right) = o(1/\varphi(n-1)) \\ &= o(1/\varphi(n)), \qquad n \to \infty, \end{aligned}$$

which contradicts (4.39).

We finally remark that conditions (4.37) and (4.38) are only slight restrictions. They are satisfied by the examples treated in Remarks 2, 3, and 5, and they are implied by the assumptions of Theorem 4. Hence (4.12) may be replaced by $\|p_n^0(f)\|_Y \approx (g'(n)/g^{*\prime}(n))(\varphi^*(n)/\varphi(n))$, $n \to \infty$, and this is how far an improvement of Zamansky's inequality can go in general spaces $X$. For particular spaces $X$, however, it may happen that this extremal order is not attained (cf. [8, 9] for the dependence on $p$ of the improving factor in case $X = L_{2\pi}^p$, $p \geqslant 1$, if $\varphi^*$, $\varphi$ are exponential orders).

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